ON THE INTRODUCTION OF CONVERGENCE FACTORS INTO SUMMABLE SERIES AND SUMMABLE INTEGRALS*

BY

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The object of this paper is to develop certain general theorems about convergence factors. In the case of series we shall mean by convergence factors a set of functions of a parameter which, when introduced as factors of the successive terms of the series, cause a divergent series to converge, or a series which is already convergent to converge more rapidly, throughout a certain range of values of the parameter. In the case of integrals we shall mean by a convergence factor a function of the variable of integration and a new parameter which, when introduced as a factor of the integrand, causes a divergent integral to converge, or an integral which is already convergent to converge more rapidly, throughout a certain range of values of the parameter.

Although the name convergence factor is of recent origin, the subject itself, in the simple case of a convergent series, goes back to ABEL and virtually takes its rise in his well known theorem on the continuity of a power series. The successive powers of x in the terms of a power series are, in fact, convergence factors of a simple nature, though not ordinarily regarded as such.

The first attempt to extend the theory of convergence factors to general types of divergent series was made by FROBENIUS.† The class of series which he considered will be designated in this paper as summable series ‡ and may be defined as follows: Given a series

$$u_0 + u_1 + u_2 + \cdots,$$

we represent by s_n the sum of its first $(n+1)$ terms and form

$$S_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}.$$

If the limit

$$\lim_{n=\infty} S_n$$

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† Crelle's Journal, vol. 89 (1880), p. 262.

‡ Many authors use this term in a more general sense.

exists, we say that the series is summable and attach to it the value of that limit. It is a well known and easily established theorem that every convergent series is summable, and that the process of summation leads to the value of the series as ordinarily defined.

The theorem proved by FROBENIUS in the article above mentioned includes only the simple class of convergence factors first considered by ABEL, i. e., successive powers of the parameter. A more general theorem was proved by FEJÉR,* in which the convergence factors have the form $\phi(0), \phi(t), \phi(2t), \cdots$. A similar theorem concerning still more general convergence factors was stated without proof by HARDY at the close of a recent paper in the Proceedings of the London Mathematical Society.† The present investigation had been begun and the essential results of §§ 1 and 2 obtained and communicated to Professor BÔCHER before the appearance of HARDY's paper.

The principal results of the present paper are contained in Theorems I, II, IV, and V. The method of proof in these four theorems is a development of that used by RIEMANN in proving a certain lemma (cf. RIEMANN, *Mathematische Werke*, 2d ed., p. 246). This lemma appears as a special case of Theorem II if we take $f_n(\alpha) = \sin^2 n\alpha / n^2 \alpha^2$.

§ 1. *Convergence factors in summable series.*

THEOREM I. *If the series*

$$u_0 + u_1 + u_2 + \cdots$$

is summable and has the value S, then the series

$$F(\alpha) = u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots \tag{1}$$

will be absolutely convergent and continuous for all positive values of α, and will approach S as its limit when $\alpha = +0$, provided the convergence factors $f_1(\alpha), f_2(\alpha), \cdots$ satisfy the following conditions:

$$(a) \qquad f_n(\alpha) \text{ is continuous} \qquad (\alpha \geqq 0),$$

$$(b) \qquad |f_n(\alpha)| < \frac{N}{n^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

$$(c) \qquad f_n(0) = 1,$$

* Mathematische Annalen, vol. 58 (1904), p. 62.

† Ser. 2, vol. 4 (1906), part 4; p. 247. HARDY'S results are somewhat similar to Theorems I and II below, but the conditions imposed on the convergence factors are decidedly different, and his theorems are on the whole less far reaching. His requirement, in the case of summable series, that the differences between two successive convergence factors be positive, will, I think, be found unnecessary if a method of proof analogous to the proof of Theorem I of this article be followed.

$$(d) \qquad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \geqq 0\,^* \qquad (0 \leqq \alpha \leqq c;\ 0 \leqq n\alpha \leqq c;\ n = 0, 1, \cdots),^\dagger$$

$$(e) \qquad |f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{L}{n^{2+\rho}\alpha^\rho} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive the following further condition which is satisfied by the convergence factors,

$$(f) \qquad |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{L_1}{n^{1+\rho}\alpha^\rho} \qquad (\alpha > 0),$$

where L_1 is a positive constant.

We have from condition (b)

$$\lim_{n=\infty} [f_n(\alpha) - f_{n+1}(\alpha)] = 0 \qquad (\alpha > 0),$$

and consequently

$$f_n(\alpha) - f_{n+1}(\alpha) = \sum_{n=n}^{\infty} \{ [f_n(\alpha) - f_{n+1}(\alpha)] - [f_{n+1}(\alpha) - f_{n+2}(\alpha)] \}$$
$$= \sum_{n=n}^{\infty} \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \}.$$

Hence by virtue of condition (e)

$$(2) \qquad \begin{aligned} |f_n(\alpha) - f_{n+1}(\alpha)| &< \sum_{n=n}^{\infty} \frac{L}{n^{2+\rho}\alpha^\rho} < \int_{n-1}^{\infty} \frac{L}{x^{2+\rho}\alpha^\rho}\,dx = \frac{L}{(1+\rho)(n-1)^{1+\rho}\alpha^\rho} \\ &= \frac{\dfrac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho}\left(\frac{n}{n-1}\right)^{1+\rho} < \frac{2^{1+\rho}\dfrac{L}{1+\rho}}{n^{1+\rho}\alpha^\rho} \qquad (\alpha > 0;\ n \geqq 2). \end{aligned}$$

For $n = 1$ we have from condition (b)

$$(3) \qquad |f_1(\alpha) - f_2(\alpha)| < \frac{2N}{\alpha^{2+\rho}} \leqq \frac{2N}{\alpha^{1+\rho}} \qquad (\alpha \geqq 1),$$

and from condition (a)

$$(4) \qquad |f_1(\alpha) - f_2(\alpha)| < \frac{M}{\alpha^{1+\rho}} \qquad (0 < \alpha < 1),$$

where M is the greatest numerical value of $[f_1(\alpha) - f_2(\alpha)]$ in the interval $0 \leqq \alpha \leqq 1$. Choosing as L_1 the greatest of the thre equantities M, $2N$, and $2^{1+\rho}L/(1+\rho)$ we get (f) by combining (2), (3), and (4).

We shall next prove that the series (1) is absolutely convergent for every positive value of α. Let

$$s_n = \sum_{n=0}^{n} u_n$$

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* We may substitute for the inequality in (d) the condition

$$(d') \qquad f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \leqq 0.$$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† For $n = 0$ we take here $f_0(\alpha) = 1$.

$$S_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

If for the sake of uniformity we let $S_{-2} = S_{-1} = 0$, we have

$$u_n = (n+1) S_n - 2n S_{n-1} + (n-1) S_{n-2} \qquad (n = 0, 1, 2, \cdots), \tag{5}$$

and consequently

$$\frac{u_n}{n} = \left(1 + \frac{1}{n}\right) S_n - 2 S_{n-1} + \left(1 - \frac{1}{n}\right) S_{n-2} \qquad (n = 1, 2, \cdots),$$

whence it follows that

$$\lim_{n=\infty} \left(\frac{u_n}{n}\right) = 0.$$

We can, therefore, take k so that

$$\left|\frac{u_n}{n}\right| < k \qquad (n \geqq 1).$$

Combining this with condition (b) we see that the terms of (1) are, if we omit the first, less in absolute value than the corresponding terms of the series

$$\sum_{n=1}^{\infty} \frac{Nk}{\alpha^{2+\rho} n^{1+\rho}}.$$

Since this last series is convergent, the absolute convergence of (1) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We may by virtue of (5) write

$$F(\alpha) = \sum_{n=0}^{\infty} \{(n+1) S_n - 2n S_{n-1} + (n-1) S_{n-2}\} f_n(\alpha). \tag{6}$$

In the series (6) we have a right to leave out the parentheses, since it is convergent, since each parenthesis contains only three terms, and since by condition (b) the general term of the resulting series has the limit zero. Moreover, in this latter series we have a right to rearrange the terms so that terms involving the same S are grouped together, since in the rearrangement we do not carry any term over more than three terms. Hence

$$F(\alpha) = \sum_{n=0}^{\infty} (n+1) S_n \{f_n(\alpha) - 2 f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \tag{7}$$

Placing

$$S_n = S + \epsilon_n \tag{8}$$

we have

$$\begin{aligned} F(\alpha) = {} & S \sum_{n=0}^{\infty} (n+1) \{f_n(\alpha) - 2 f_{n+1}(\alpha) + f_{n+2}(\alpha)\} \\ & + \sum_{n=0}^{\infty} \epsilon_n (n+1) \{f_n(\alpha) - 2 f_{n+1}(\alpha) + f_{n+2}(\alpha)\}. \end{aligned} \tag{9}$$

But

$$\sum_{n=0}^{n}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1-(n+2)f_{n+1}(\alpha)+(n+1)f_{n+2}(\alpha).$$

Whence, by condition (b),

$$\sum_{n=0}^{\infty}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=1 \qquad (\alpha>0),$$

and (9) reduces to

$$(10)\qquad F(\alpha)-S=\sum_{n=0}^{\infty}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}.$$

By (c) the series on the right hand side of (10) converges when $\alpha=0$ and has the value 0. Accordingly, if we can show that it is uniformly convergent for values of $\alpha \geqq 0$ our theorem will be proved, since by condition (a) each term is a continuous function of α.

Given a positive quantity δ, arbitrarily small, we wish to show that we can choose q so large that

$$(11)\qquad \left|\sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}\right|<\delta \quad (\nu\geqq\mu\geqq q,\ \alpha\geqq 0)$$

Take $m>2$ and such that

$$(12)\qquad |\epsilon_n|<\eta=\frac{\rho c^{2+\rho}\,\delta}{\rho c^{2+\rho}+\rho N+\rho c^2 L_1+2^{1+\rho}c^2 L} \qquad (n\geqq m).$$

We first consider values of α in the interval $0<\alpha<c/2$, and represent by s the greatest integer less than c/α. Let μ and ν be any two integers such that

$$m\leqq\mu\leqq\nu.$$

Three cases must be considered:

(A) $\qquad \mu\leqq s<\nu,$

(B) $\qquad \nu\leqq s,$

(C) $\qquad \mu>s.$

Beginning with case (A) we write

$$(13)\quad \sum_{n=\mu}^{n=\nu}\epsilon_n(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}=\sum_{\mu}^{s}+\sum_{s+1}^{\nu}=R_1+R_2.$$

By virtue of (d)* and (12) we have for R_1

$$(14)\quad |R_1|<\eta\sum_{0}^{s}(n+1)\{f_n(\alpha)-2f_{n+1}(\alpha)+f_{n+2}(\alpha)\}$$
$$=\eta\{1-f_{s+1}(\alpha)-(s+1)[f_{s+1}(\alpha)-f_{s+2}(\alpha)]\}.$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

From condition (b) we get

$$(15) \qquad |f_{s+1}(\alpha)| < \frac{N}{[(s+1)\alpha]^{2+\rho}} \leqq \frac{N}{c^{2+\rho}},$$

and from condition (f)

$$(16) \qquad (s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)| < \frac{L_1}{[(s+1)\alpha]^{\rho}} \leqq \frac{L_1}{c^{\rho}}.$$

By combining (14), (15), and (16) we obtain

$$(17) \qquad |R_1| < \eta \left\{ 1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^{\rho}} \right\}.$$

According to condition (e) we have for R_2

$$|R_2| < \eta \sum_{s+1}^{\nu} (n+1)|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < \frac{\eta L}{\alpha^{\rho}} \sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}}.$$

Furthermore

$$\sum_{s+1}^{\nu} \frac{(n+1)}{n^{2+\rho}} < \int_s^{\nu} \frac{x+1}{x^{2+\rho}}\, dx = \left[-\frac{1}{\rho x^{\rho}} - \frac{1}{(1+\rho)x^{1+\rho}} \right]_s^{\nu} < \left[\frac{1}{\rho s^{\rho}} + \frac{1}{(1+\rho)s^{1+\rho}} \right] < \frac{2}{\rho s^{\rho}}.$$

Hence

$$|R_2| < \eta \frac{2L}{\rho(s\alpha)^{\rho}}.$$

But

$$s \geqq \frac{c}{\alpha} - 1$$

or

$$s\alpha \geqq c - \alpha > c - \frac{c}{2} = \frac{c}{2},$$

so that we have finally for R_2,

$$(18) \qquad |R_2| < \eta \frac{2^{1+\rho} L}{\rho c^{\rho}}.$$

Combining (13), (17), and (18) we get

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \} \right|$$
$$< \eta \frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1 + 2^{1+\rho} c^2 L}{\rho c^{2+\rho}} = \delta.$$

We consider next case (B). We have

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \} \right|$$
$$\leqq \eta \sum_{0}^{s} (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \}$$

$$= \eta \{ 1 - f_{s+1}(\alpha) - (s+1)[f_{s+1}(\alpha) - f_{s+2}(\alpha)] \}$$
$$\leqq \eta \{ 1 + |f_{s+1}(\alpha)| + (s+1)|f_{s+1}(\alpha) - f_{s+2}(\alpha)| \}$$
$$< \eta \left\{ 1 + \frac{N}{c^{2+\rho}} + \frac{L_1}{c^\rho} \right\} = \eta \frac{\rho c^{2+\rho} + \rho N + \rho c^2 L_1}{\rho c^{2+\rho}} < \delta.$$

For case (C) we have

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \} \right|$$
$$\leqq \eta \sum_{s+1}^{\nu} (n+1) | f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) |$$
$$< \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{n+1}{n^{2+\rho}} < \eta \frac{2^{1+\rho} L}{\rho c^\rho} = \eta \frac{2^{1+\rho} c^2 L}{\rho c^{2+\rho}} < \delta.$$

Thus we have proved (11) in all three cases for $0 < \alpha < c/2$, the value of q being equal to m.

For values of $\alpha \geqq c/2$ we have from condition (e)

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \} \right|$$
$$\leqq \sum_{\mu}^{\nu} | \epsilon_n | \frac{\left(\frac{2}{c}\right)^\rho (n+1) L}{n^{2+\rho}} \leqq \left(\frac{2}{c}\right)^\rho LK \sum_{\mu}^{\nu} \frac{(n+1)}{n^{2+\rho}}$$

where K is the greatest numerical value of the quantities $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$. Since the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^{2+\rho}}$$

is convergent, we can find m_1 such that

$$\left| \sum_{n=\mu}^{n=\nu} \epsilon_n (n+1) \{ f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \} \right| < \delta \qquad \left(\begin{matrix} \nu \geqq \mu \geqq m_1 \\ \alpha \geqq c/2 \end{matrix} \right).$$

For $\alpha = 0$ the inequality (11) is obviously satisfied for all values of q since each $f_i(\alpha) = 1$. If, therefore, we choose for q the greater of the two quantities m and m_1, it will hold for all values of $\alpha \geqq 0$, as was to be proved.

Convergence factors of the form $\phi(n\alpha)$ are an important class, and it is interesting to note what restrictions may be placed on $\phi(x)$ in order that the conditions of Theorem I may be satisfied. We will show that if

(i) $$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x > 0),$$

(ii) $$\phi(0) = 1,$$

(iii) $$\phi''(x) \text{ exists and } \phi''(x) \geqq 0\,^* \qquad (0 \leqq x \leqq c),$$

(iv) $$|\phi''(x)| < \frac{L}{x^{2+\rho}} \qquad (x > 0),$$

then the functions $\phi(0) = 1$, $\phi(\alpha)$, $\phi(2\alpha)$, $\cdots$ satisfy the conditions of Theorem I.

Condition (a) follows from the fact that the second derivative of $\phi(x)$ exists for all values of $x \geqq 0$. Conditions (b) and (c) follow at once from (i) and (ii).

We have (cf. MARKOFF, *Differenzenrechnung*, § 8)

(19) $$\phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha] = \alpha^2 \phi''[(n+\theta)\alpha] \qquad (0 < \theta < 2).$$

Hence by virtue of (iii) condition (d) will be fulfilled, if c is there replaced by $c/3$.

Finally, by (19) and (iv)

$$|\phi(n\alpha) - 2\phi[(n+1)\alpha] + \phi[(n+2)\alpha]| < \frac{L}{n^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0;\ n = 1, 2, \cdots),$$

and (e) is satisfied.

The theorem of FEJÉR mentioned in the introduction, in which the convergence factors are functions of $n\alpha$, is not, in its greatest generality, a special case of Theorem I, but becomes so if we exclude from it all functions whose first derivatives have an infinite number of maxima and minima in the neighborhood of the origin. I shall show this by proving that if we have a function $\phi(x)$ which satisfies the conditions of FEJÉR's theorem and whose first derivative has not an infinite number of maxima and minima in the neighborhood of the origin, then it will satisfy (i)–(iv).

FEJÉR's conditions are

(a) $$|\phi(x)| < \frac{M}{x^{2+\rho}} \qquad (x > 1),$$

(b) $$|\phi''(x)| < \frac{M}{x^{2+\rho}} \qquad (x > 1),$$

(c) $$\phi(0) = 1,$$

(d) $$|\phi''(x)| < \mu\,^\dagger \qquad (0 \leqq x \leqq 1),$$

where M, ρ and μ are positive constants.

From (d) it follows that $\phi(x)$ is continuous in the interval $0 \leqq x \leqq 1$. We can therefore choose a positive constant N greater than M and greater than the

* We may substitute for (iii)

(iii′) $$\phi''(x) \leqq 0 \qquad (0 \leqq x \leqq c).$$

† (d) is not explicitly stated as a condition, but is used in the course of the proof.

greatest numerical value of $\phi(x)$ in the interval $0 \leqq x \leqq 1$, so that we have

$$|\phi(x)| < \frac{N}{x^{2+\rho}} \qquad (x > 0),$$

and consequently (i) is satisfied. Condition (ii) is identical with (c). Since $\phi'(x)$ has not an infinite number of maxima and minima in the neighborhood of the origin, either (iii) or (iii') must be satisfied. By taking for L the greater of the two quantities M and μ we have in view of (b) and (d)

$$|\phi''(x)| < \frac{L}{x^{2+\rho}} \qquad (x > 0),$$

and (iv) is satisfied.

Thus we see that Theorem I, when applied to functions of $n\alpha$, includes all functions satisfying FEJÉR's conditions that are likely to be useful as convergence factors. In addition to this it includes an important class of functions that are not included in FEJÉR's theorem, i. e., functions that satisfy (i)–(iv) but whose second derivative becomes infinite in the neighborhood of the origin. An example of such a function is $e^{-\sqrt{x}}$.

A general theorem which includes FEJÉR's theorem as a special case but does not include functions of the type just mentioned is obtained by replacing condition (d) of Theorem I by the condition

$$|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < K\alpha^2 \qquad (0 \leqq n\alpha \leqq c).$$

The proof is quite similar to the proof of Theorem I.

§ 2. *Convergence factors in convergent series.*

Theorem I is evidently applicable to convergent series since every convergent series is summable. When dealing with convergent series, however, we do not need to place as much restriction upon the convergence factors as in the general case of a summable series. This is shown by the following theorem which in many respects is more general than Theorem I:

THEOREM II. *If the series*

$$(20) \qquad u_0 + u_1 + u_2 + \cdots$$

converges to the value A, then the series

$$(21) \qquad u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \cdots$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factors *

* That the series (21) converges more rapidly than (20) appears from condition (b).

$f_1(\alpha)$, $f_2(\alpha)$, $\cdots$ *satisfy the following conditions:*

$$(a) \qquad f_n(\alpha) \text{ is continuous} \qquad (\alpha \geqq 0),$$

$$(b) \qquad |f_n(\alpha)| < \frac{N}{n^{1+\rho}\alpha^{1+\rho}} \qquad (\alpha > 0),$$

$$(c) \qquad f_n(0) = 1,$$

$$(d) \qquad f_n(\alpha) - f_{n+1}(\alpha) \geqq 0\,^* \qquad (n \geqq 1,\ 0 \leqq n\alpha \leqq c),$$

$$(e) \qquad |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{L}{n^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c *and* L *are positive constants.*

We must first derive from the given conditions the following further condition:

$$(f) \qquad |f_n(\alpha)| < K \qquad (\alpha \geqq 0)$$

where K is a positive constant.

We know from condition (d) † that, for a given value of α less than c, the functions $f_n(\alpha)$ form a sequence that never increases as long as $n\alpha$ does not exceed c. For each value of $\alpha < c$ we pick out the greatest integer $r \leqq c/\alpha$. This r satisfies the inequality

$$(22) \qquad \frac{c}{2} < r\alpha \leqq c.$$

We have furthermore

$$(23) \qquad f_1(\alpha) \geqq f_n(\alpha) \geqq f_r(\alpha) \qquad (0 < n\alpha < c).$$

But by condition (a) we can find a positive constant M such that

$$|f_1(\alpha)| < M \qquad (0 < \alpha < c),$$

and from (b) and (22)

$$|f_r(\alpha)| < \frac{N}{(r\alpha)^{1+\rho}} < \frac{2^{1+\rho}N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $2^{1+\rho}N/c^{1+\rho}$, we have by (23)

$$|f_n(\alpha)| < K_1 \qquad (0 < n\alpha < c).$$

For $n\alpha \geqq c$ we have from condition (b)

$$|f_n(\alpha)| < \frac{N}{(n\alpha)^{1+\rho}} \leqq \frac{N}{c^{1+\rho}} < K_1 \qquad (n\alpha \geqq c).$$

* This is the only condition that is not satisfied by the convergence factors of Theorem I. We may substitute for it the condition

$$(d') \qquad f_n(\alpha) - f_{n+1}(\alpha) \leqq 0 \qquad (0 \leqq n\alpha \leqq c).$$

If (d) (or (d')) holds for all values of n and α, (e) is unnecessary.

† If we use (d') instead of (d), the inequality signs in (22) are reversed, but this does not affect the reasoning.

For $\alpha = 0$ we have from condition (c)

$$f_n(0) = 1.$$

Hence if we choose K greater than either of the two quantities K_1 and 1, we have the desired condition (f). Place now

$$u_0 + u_1 + u_2 + \cdots + u_{n-1} = A + \epsilon_n$$

so that

$$\lim_{n=\infty} \epsilon_n = 0,$$

and let δ be an arbitrarily small, positive quantity. Since by condition (a) each term of (21) is a continuous function of α, our theorem will be proved if we can show that a positive integer q exists for which

$$\left|\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha)\right| < \delta \qquad (\nu \geqq \mu \geqq q;\ \alpha \geqq 0). \tag{24}$$

Determine a value of $m > 2$ such that

$$|\epsilon_n| < \eta = \frac{\rho c^\rho \delta}{4\rho c^\rho K + 2^\rho L} \qquad (n \geqq m). \tag{25}$$

We first consider values of α in the interval $0 < \alpha < c/2$. Then

$$\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha) = \sum_{\mu}^{\nu} (\epsilon_{n+1} - \epsilon_n) f_n(\alpha) \tag{26}$$

$$= -\epsilon_\mu f_\mu(\alpha) + \sum_{\mu}^{\nu} \epsilon_{n+1}[f_n(\alpha) - f_{n+1}(\alpha)] + \epsilon_{\nu+1} f_{\nu+1}(\alpha).$$

Let s be the greatest integer less than c/α, and μ and ν any two integers for which

$$m \leqq \mu \leqq \nu.$$

Three cases must be considered:

(A) $\qquad \mu \leqq s < \nu,$

(B) $\qquad \nu \leqq s,$

(C) $\qquad \mu > s.$

Beginning with (A) we put

$$\sum_{n=\mu}^{n=\nu} \epsilon_{n+1}[f_n(\alpha) - f_{n+1}(\alpha)] = \sum_{\mu}^{s} + \sum_{s}^{\nu} = R_1 + R_2. \tag{27}$$

By virtue of (d),* (f), and (25) we have for R_1

$$
\begin{aligned}
|R_1| &\leqq \sum_{\mu}^{s} |\epsilon_{n+1}|[f_n(\alpha) - f_{n+1}(\alpha)] \\
&< \eta \sum_{\mu}^{s} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta\,[f_\mu(\alpha) - f_{s+1}(\alpha)] < 2K\eta.
\end{aligned}
\tag{28}
$$

For R_2 we have by condition (e)

$$
\begin{aligned}
|R_2| &\leqq \sum_{s+1}^{\nu} |\epsilon_{n+1}\,[f_n(\alpha) - f_{n+1}(\alpha)]| \\
&< \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \frac{\eta L}{\alpha^\rho} \int_s^\nu \frac{dx}{x^{1+\rho}} = \frac{\eta L}{\alpha^\rho}\left[\frac{-1}{\rho x^\rho}\right]_s^\nu < \frac{\eta L}{\rho s^\rho \alpha^\rho}.
\end{aligned}
$$

But

$$
s\alpha \geqq \left(\frac{c}{\alpha} - 1\right)\alpha = c - \alpha > c - \frac{c}{2} = \frac{c}{2},
$$

and hence

$$
|R_2| < \frac{\eta 2^\rho L}{\rho c^\rho}. \tag{29}
$$

Combining (26), (27), (28), and (29) we get

$$
\begin{aligned}
\left|\sum_{\mu}^{\nu} u_n f_n(\alpha)\right| &\leqq |\epsilon_\mu f_\mu(\alpha)| + |R_1| + |R_2| + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| \\
&< K\eta + 2K\eta + \frac{\eta 2^\rho L}{\rho c^\rho} + K\eta = \eta\left[\frac{4\rho c^\rho K + 2^\rho L}{\rho c^\rho}\right] = \delta.
\end{aligned}
$$

We consider next case (B). We have

$$
\begin{aligned}
&\left|\sum_{\mu}^{\nu} \epsilon_{n+1}\,[f_n(\alpha) - f_{n+1}(\alpha)]\right| \\
&\qquad < \eta \sum_{\mu}^{\nu} [f_n(\alpha) - f_{n+1}(\alpha)] = \eta\,[f_\mu(\alpha) - f_{\nu+1}(\alpha)] < 2K\eta.
\end{aligned}
$$

Combining this result with (26) we get

$$
\left|\sum_{\mu}^{\nu} u_n f_n(\alpha)\right| < |\epsilon_\mu f_\mu(\alpha)| + 2K\eta + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < 4K\eta < \delta.
$$

For case (C) we have

$$
\left|\sum_{\mu}^{\nu} \epsilon_{n+1}\,[f_n(\alpha) - f_{n+1}(\alpha)]\right| < \eta \sum_{s+1}^{\nu} |f_n(\alpha) - f_{n+1}(\alpha)| < \frac{\eta L}{\alpha^\rho} \sum_{s+1}^{\nu} \frac{1}{n^{1+\rho}} < \eta\,\frac{2^\rho L}{\rho c^\rho},
$$

which combined with (26) gives

$$
\left|\sum_{\mu}^{\nu} u_n f_n(\alpha)\right| < |\epsilon_\mu f_\mu(\alpha)| + \eta\,\frac{2^\rho L}{\rho c^\rho} + |\epsilon_{\nu+1} f_{\nu+1}(\alpha)| < K\eta + \eta\,\frac{2^\rho L}{\rho c^\rho} + K\eta < \delta.
$$

* The substitution of (d') for (d) occasions only a slight modification of the proof.

Hence the desired inequality (24) is satisfied in all three cases, q being equal to m.

For all values of $\alpha \geqq c/2$ we have

$$|u_n f_n(\alpha)| < \frac{BN}{n^{1+\rho}\alpha^{1+\rho}} \leqq \frac{BN2^{1+\rho}}{n^{1+\rho}c^{1+\rho}}$$

where B is the largest numerical value of the terms $u_0, u_1, u_2, \cdots$. The series $\sum 1/n^{1+\rho}$ converges, and therefore we can choose m_1 such that

$$\left|\sum_{n=\mu}^{n=\nu} u_n f_n(\alpha)\right| < \delta \qquad \left(\nu \geqq \mu \geqq m_1;\ \alpha \geqq \frac{c}{2}\right).$$

When $\alpha = 0$ the series (21) becomes the series (20), and we can so choose m_2 that

$$\left|\sum_{n=\mu}^{n=\nu} u_n\right| < \delta \qquad (\nu \geqq \mu \geqq m_2).$$

Thus, finally, if we take for q the greatest of the three values m, m_1, and m_2, we have the desired inequality (24).

The convergence factors $e^{-n\alpha}$ satisfy the conditions of Theorem II. Hence if the series

$$a_0 + a_1 r + a_2 r^2 + \cdots$$

is convergent, the series

$$a_0 + a_1 r e^{-\alpha} + a_2 r^2 e^{-2\alpha} + \cdots$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set $e^{-\alpha} = x/r$, it follows that

$$a_0 + a_1 r\left(\frac{x}{r}\right) + a_2 r^2 \left(\frac{x}{r}\right)^2 + \cdots$$

or

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

converges uniformly in the interval $0 \leqq x \leqq r$. Hence Theorem II includes ABEL's theorem as a special case.

§ 3. *Summable integrals. General facts and lemmas.*

Let $f(x)$ be a function which is integrable in every finite interval lying in the interval $a \leqq x$. Then the integral

$$\int_a^\infty f(x)\,dx$$

shall be said to be summable if the limit

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)\,d\beta d\alpha\right]$$

exists; and we attach to the integral the value of that limit. The agreement of this definition with the ordinary definition of an integral, is shown by the following theorem:

THEOREM III. *Every convergent integral is summable, and the process of summation gives the value of the integral ordinarily defined.**

Put

$$\int_a^\infty f(x)\,dx = A$$

and

$$\int_a^x f(x)\,dx = A + \epsilon(x),$$

so that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

Then

$$\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)\,d\beta\,d\alpha = \frac{1}{x}\int_a^x [A + \epsilon(\alpha)]\,d\alpha = A - \frac{aA}{x} + \frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha.$$

Since the second term approaches zero as x becomes infinite, our theorem will be proved if we can show that the same is true of the third term. Let δ be a positive constant as small as we please, and choose m so that

$$|\epsilon(\alpha)| < \frac{\delta}{2} \qquad (\alpha \geqq m > 0).$$

Then for values of $x \geqq m$ we have

$$\left|\frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha\right| \leqq \frac{1}{x}\left|\int_a^m \epsilon(\alpha)\,d\alpha\right| + \frac{1}{x}\left|\int_m^x \epsilon(\alpha)\,d\alpha\right|.$$

The second of these terms reduces to

$$\frac{x-m}{x}|\epsilon(\alpha')| \qquad (m < \alpha' < x)$$

and is therefore less than $\delta/2$. We can also choose p so that when $x > p$ the first term is less than $\delta/2$. Accordingly, since δ is arbitrary,

$$\lim_{x=\infty}\left[\frac{1}{x}\int_a^x \epsilon(\alpha)\,d\alpha\right] = 0,$$

and our theorem is proved.

I shall next establish a series of lemmas that will be needed in the proof of the theorem concerning the introduction of convergence factors into summable integrals.

---

* The substance of this theorem is stated without proof by HARDY in an article in the Quarterly Journal of Mathematics, vol. 35 (1903–04), p. 54.

LEMMA 1. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then there exists a positive constant M such that*

$$|f(x)| < Mx \qquad (x \geqq k).$$

Let us choose δ such that

$$(30) \qquad |f(x+h) - f(x)| < 1 \qquad (|h| \leqq \delta;\ x \geqq k).$$

Then for every $x \geqq k$ we can find a $y \geqq 0$ such that

$$x = k + y\delta.$$

If n is the greatest integer not exceeding y, we have by (30)

$$|f(k+y\delta) - f(k)| < n+1 \qquad (y \geqq 0),$$

and therefore

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k+y\delta} + \frac{n+1}{k+y\delta}.$$

For values of y in the interval $0 \leqq y < 1$ we have, since $n = 0$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{1}{k} = K_1.$$

For values of $y \geqq 1$, since $y \geqq n \geqq 1$,

$$\frac{|f(k+y\delta)|}{k+y\delta} < \frac{|f(k)|}{k} + \frac{n+1}{n\delta} \leqq \frac{|f(k)|}{k} + \frac{2}{\delta} = K_2.$$

If then we choose as M the larger of the two constants K_1 and K_2, we have

$$|f(k+y\delta)| < M(k+y\delta) \qquad (y \geqq 0),$$

or

$$|f(x)| < Mx \qquad (x \geqq k).$$

LEMMA 2. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, then $f(x)/x^\rho$ (where ρ is any positive constant) is uniformly continuous in the same interval.*

We can assume without loss of generality that $k \geqq e$, where e is the base of natural logarithms; for, if it is not, it follows that $f(x)/x^\rho$ is uniformly continuous in the closed interval $k \leqq x \leqq e$ since it is continuous in it.

From Lemma 1 it follows that there exists a positive constant M such that

$$\left|\frac{f(x)}{x}\right| < M \qquad (x \geqq k).$$

Given ϵ positive and arbitrarily small, let us choose δ less than $\epsilon/2M$ and such that

$$|f(x+h) - f(x)| < \frac{k^\rho}{2}\epsilon \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

We have

$$(31)\quad \left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| \leqq \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| + |f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right|.$$

For the first term on the right hand side we have

$$(32)\quad \frac{1}{(x+h)^\rho}\,|f(x+h)-f(x)| < \frac{1}{k^\rho}\,\frac{k^\rho \epsilon}{2} = \frac{\epsilon}{2} \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

Also from the Law of the Mean

$$\frac{1}{(x+h)^\rho} - \frac{1}{x^\rho} = \frac{-\rho h}{(x+\theta h)^{\rho+1}} \qquad (0<\theta<1),$$

and hence

$$|f(x)|\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\frac{x\rho h}{x^{\rho+1}} \leqq M\frac{\rho}{e^\rho}\,h \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

But

$$\frac{\rho}{e^\rho} < 1 \qquad (\rho > 0).$$

Consequently

$$(33)\quad |f(x)|\cdot\left|\frac{1}{(x+h)^\rho}-\frac{1}{x^\rho}\right| < M\delta < \frac{\epsilon}{2} \qquad (x \geqq k;\ 0 < h \leqq \delta).$$

Combining (31), (32), and (33) we have

$$\left|\frac{f(x+h)}{(x+h)^\rho}-\frac{f(x)}{x^\rho}\right| < \epsilon \qquad (x \geqq k;\ 0 < h \leqq \delta),$$

and the lemma is proved.

Lemma 3. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$, and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0.$$

We know from Lemma 2 that $f(x)/x$ is uniformly continuous for values of $x \geqq k$. Given ϵ, let us choose a positive $\delta < k$ and such that

$$(34)\quad \left|\frac{f(x+h)}{x+h}-\frac{f(x)}{x}\right| < \frac{\epsilon}{3} \qquad (x \geqq k;\ 0 < h \leqq 2\delta).$$

Let

$$\frac{1}{x}\int_a^x \int_a^\alpha f(\beta)\,d\beta\, d\alpha = S(x).$$

Then

$$(35)\qquad f(x)=\frac{d^2[xS(x)]}{dx^2}.$$

Moreover, since $\lim_{x=\infty} S(x)$ exists, we can so determine $x_0 \geqq k$ that

$$(36)\qquad |S(x')-S(x)|<\frac{\delta^2\epsilon}{6} \qquad (x'>x\geqq x_0).$$

But we have *

$$(x+2\delta)S(x+2\delta)-2(x+\delta)S(x+\delta)+xS(x)=\delta^2 f(x+\theta\delta) \quad (0<\theta<2),$$

whence by (36)

$$\delta^2|f(x+\theta\delta)|<(x+2\delta)|S(x+2\delta)-S(x+\delta)| \\ +x|S(x+\delta)-S(x)|<\frac{2(x+\delta)\delta^2\epsilon}{6} \quad (x\geqq x_0).$$

Consequently, since $x+\delta<2x$, we get

$$\left|\frac{f(x+\theta\delta)}{x+\theta\delta}\right|<\frac{2(x+\delta)}{(x+\theta\delta)}\frac{\epsilon}{6}<\frac{2(x+\delta)}{x}\frac{\epsilon}{6}<\frac{2\epsilon}{3} \qquad (x\geqq x_0).$$

It follows then from (34) that

$$\left|\frac{f(x)}{x}\right|<\epsilon \qquad (x\geqq x_0),$$

and therefore

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right]=0.$$

Before taking up the other lemmas, I wish to introduce the following notation. Let

$$(37)\qquad \int f(x)\,dx=\psi(x)+C_1, \qquad \int\psi(x)\,dx=\chi(x)+C_2.$$

Then

$$(38)\qquad xS(x)=\int_a^x\int_a^\alpha f(\beta)\,d\beta\,d\alpha=\int_a^x[\psi(\alpha)-\psi(a)]\,d\alpha \\ =\chi(x)-x\psi(a)-\chi(a)+a\psi(a).$$

LEMMA 4. *If $f(x)$ is uniformly continuous for values of $x\geqq k>0$, and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right]=0.$$

From Lemma 3 it follows that for a given positive but arbitrarily small ϵ, we

* Cf. MARKOFF: *Differenzenrechnung*, § 8.

can choose a positive $x_0 > a$ such that

$$\left|\frac{f(x)}{x}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

We have also from (37)

$$\psi(x) - \psi(x_0) = (x - x_0) f[x_0 + \theta(x - x_0)] \qquad (0 < \theta < 1),$$

and consequently

$$\left|\frac{\psi(x)}{x^2}\right| \leqq \left|\frac{\psi(x_0)}{x^2}\right| + \frac{x - x_0}{x} \cdot \left|\frac{f[x_0 + \theta(x - x_0)]}{x_0 + \theta(x - x_0)}\right| \cdot \frac{x_0 + \theta(x - x_0)}{x}$$

$$< \left|\frac{\psi(x_0)}{x^2}\right| + \frac{\epsilon}{2} \qquad (x \geqq x_0).$$

Choose $x_1 \geqq x_0$ and such that

$$\left|\frac{\psi(x_0)}{x^2}\right| < \frac{\epsilon}{2} \qquad (x \geqq x_1).$$

Then

$$\left|\frac{\psi(x)}{x^2}\right| < \epsilon \qquad (x \geqq x_1),$$

and accordingly

$$\lim_{x=\infty} \left[\frac{\psi(x)}{x^2}\right] = 0.$$

LEMMA 5. *If we have a function $\phi(x)$ such that*

$$\lim_{x=\infty} [\phi(x)] = C, \qquad \lim_{x=\infty} [\phi''(x)] = 0$$

where C is any constant, then it follows that

$$\lim_{x=\infty} [\phi'(x)] = 0.$$

Given ϵ positive and arbitrarily small, let us choose B such that

$$(39) \qquad |\phi(x') - \phi(x)| < \frac{\epsilon}{2}, \qquad |\phi''(x)| < \frac{\epsilon}{2} \qquad (x' > x \geqq B).$$

From the Law of the Mean

$$\phi(x + 1) - \phi(x) = \phi'(x + \theta) \qquad (0 < \theta < 1),$$

and therefore

$$(40) \qquad |\phi'(x + \theta)| < \frac{\epsilon}{2} \qquad (x \geqq B).$$

Applying again the Law of the Mean, we have

$$\phi'(x + 1) - \phi'(x + \theta) = (1 - \theta)\phi''(x + \theta') \qquad (\theta < \theta' < 1),$$

from which it follows, by (39) and (40), that

$$|\phi'(x + 1)| < |\phi'(x + \theta)| + |\phi''(x + \theta')| < \epsilon \qquad (x \geqq B).$$

Consequently

$$\lim_{x=\infty} [\phi'(x)] = 0.$$

LEMMA 6. *If $f(x)$ is uniformly continuous for values of $x \geqq k > 0$ and the integral*

$$\int_a^\infty f(x)\,dx$$

is summable, then

$$\lim_{x=\infty} [S'(x)] = 0.$$

We have from (38)

$$S''(x) = \frac{2\chi(x)}{x^3} - \frac{2\psi(x)}{x^2} + \frac{f(x)}{x} - \frac{2(\chi(a) - a\psi(a))}{x^3}. \tag{41}$$

From Lemmas 3 and 4 it follows that

$$\lim_{x=\infty}\left[\frac{f(x)}{x}\right] = 0, \qquad \lim_{x=\infty}\left[\frac{\psi(x)}{x^2}\right] = 0,$$

and from equation (38) that

$$\lim_{x=\infty}\left[\frac{\chi(x)}{x^3}\right] = 0.$$

Hence from (41)

$$\lim_{x=\infty} [S''(x)] = 0.$$

Accordingly, since by hypothesis $S(x)$ approaches a limit when $x = \infty$, we have by Lemma 5

$$\lim_{x=\infty} [S'(x)] = 0.$$

LEMMA 7. *If we have a function of two variables $\phi(\alpha, x)$ such that*

(a) $\phi(\alpha, x)$ is continuous in x and in α separately in the region $R(x_2 \geqq x \geqq x_1,\ \alpha_2 \geqq \alpha \geqq \alpha_1)$,

(b) $\phi'_x(\alpha, x)$ and $\phi''_x(\alpha, x)$ exist and

$$|\phi''_x(\alpha, x)| < C$$

in the region R, then $\phi'_x(\alpha, x)$ is continuous in α in the region R.

Take any point (α_0, x_0) that lies in the region R. Choose a constant Δx such that $x_0 + \Delta x$ lies in the interval $x_2 \geqq x \geqq x_1$ and

$$|\Delta x| < \frac{\epsilon}{4C}, \tag{42}$$

where ϵ is an arbitrarily small, positive constant.

Then choose η such that

$$(43)\qquad |\phi(\alpha', x_0+\Delta x)-\phi(\alpha_0, x_0+\Delta x)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha'-\alpha_0|<\eta),^*$$

$$(44)\qquad |\phi(\alpha', x_0)-\phi(\alpha_0, x_0)|<\frac{\epsilon}{4}\Delta x \qquad (|\alpha-\alpha_0|<\eta).$$

By the Law of the Mean

$$(45)\qquad \frac{\phi(\alpha', x_0+\Delta x)-\phi(\alpha', x_0)}{\Delta x}=\phi'_x(\alpha', x_0+\theta\Delta x) \qquad (0<\theta<1),$$

$$(46)\qquad \frac{\phi(\alpha_0, x_0+\Delta x)-\phi(\alpha_0, x_0)}{\Delta x}=\phi'_x(\alpha_0, x_0+\theta'\Delta x) \qquad (0<\theta'<1).$$

By subtraction of (46) from (45) and with the aid of (43) and (44) we obtain

$$(47)\qquad |\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha_0, x_0+\theta'\Delta x)|<\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2} \qquad (|\alpha'-\alpha_0|<\eta).$$

Applying again the Law of the Mean we have

$$\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha', x_0)=\theta\Delta x\phi''_x(\alpha', x_0+\theta_1\theta\Delta x) \qquad (0<\theta_1<1),$$

$$\phi'_x(\alpha_0, x_0+\theta'\Delta x)-\phi'_x(\alpha_0, x_0)=\theta'\Delta x\phi''_x(\alpha_0, x_0+\theta'_1\theta'\Delta x) \qquad (0<\theta'_1<1),$$

whence it follows by subtraction that

$$|\phi'_x(\alpha', x_0)-\phi'_x(\alpha_0, x_0)|\leqq|\phi'_x(\alpha', x_0+\theta\Delta x)-\phi'_x(\alpha_0, x_0+\theta'\Delta x|$$
$$+\theta'|\Delta x||\phi''_x(\alpha_0, x_0+\theta'_1\theta'\Delta x)|+\theta|\Delta x||\phi''_x(\alpha', x_0+\theta_1\theta\Delta x)|.$$

In combination with (b), (42), and (47) this gives

$$|\phi'_x(\alpha', x_0)-\phi'_x(\alpha_0, x_0)|<\epsilon \qquad (|\alpha'-\alpha_0|<\eta),$$

and the lemma is proved.

§ 4. *Convergence factors in summable integrals.*

THEOREM IV. *If the function $f(x)$ is uniformly continuous for all values of $x\geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)dx$$

is summable and has the value S, then the integral

$$(48)\qquad F(\alpha)=\int_a^\infty f(x)\phi(\alpha, x)\,dx$$

---

* It is assumed in this and in the following similar inequalities that α' lies in the interval $\alpha_2\geqq\alpha\geqq\alpha_1$.

will be absolutely convergent and continuous for all positive values of α *and will approach* S *as its limit when* $\alpha = +0$, *provided the convergence factor* $\phi(\alpha, x)$ *satisfies the following conditions for* $x \geqq a$:

$$(a) \qquad \phi(\alpha, x) \text{ is continuous in } (\alpha, x) \qquad (\alpha \geqq 0),$$

$$(b) \qquad \phi''_x(\alpha, x) \text{ exists and is continuous in } (\alpha, x) \qquad (\alpha \geqq 0),$$

$$(c) \qquad |\phi(\alpha, x)| < \frac{N}{x^{2+\rho}\alpha^{2+\rho}} \qquad (\alpha > 0),$$

$$(d) \qquad \phi(0, x) = 1,$$

$$(e) \qquad \phi''_x(\alpha, x) \geqq 0\,* \qquad (0 \leqq x\alpha \leqq c),$$

$$(f) \qquad |\phi''_x(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c, *and* L *are positive constants.*

We must first derive from the given conditions the following further conditions which hold for $x \geqq a$:

$$(g) \qquad \phi'_x(\alpha, x) \text{ is continuous in } \alpha \qquad (\alpha \geqq 0),$$

$$(h) \qquad |\phi'_x(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0).$$

Consider the function $\phi(\alpha, x)$ in the region $R(B \geqq x \geqq a;\ \alpha \geqq 0)$. By condition (a) this function is continuous throughout R. We can also show at once that its second derivative with regard to x remains finite in the region R. For from (b) we know that a positive constant M exists such that

$$|\phi''_x(\alpha, x)| < M \qquad (B \geqq x \geqq a;\ 1 \geqq \alpha \geqq 0),$$

and from condition (f) we have

$$|\phi''_x(\alpha, x)| < \frac{L}{x^{2+\rho}\alpha^{\rho}} < \frac{L}{a^{2+\rho}} \qquad (B \geqq x \geqq a;\ \alpha > 1).$$

Accordingly the function $\phi(\alpha, x)$ satisfies the conditions of Lemma 7. Consequently $\phi'_x(\alpha, x)$ is continuous in α in the region R, and therefore in the region $x \geqq a$, $\alpha \geqq 0$.

By virtue of conditions (c) and (f) we have

$$\lim_{x=\infty} [\phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

$$\lim_{x=\infty} [\phi''_x(\alpha, x)] = 0 \qquad (\alpha > 0).$$

* We may substitute for (e) the condition

$$(e') \qquad \phi''_x(\alpha, x) \leqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) (or (e')) holds for all values of x and α, (f) is unnecessary.

Hence by Lemma 5 it follows that

(49) $$\lim_{x=\infty} [\phi'_x(\alpha, x)] = 0 \qquad (\alpha > 0),$$

and consequently

$$\int_x^\infty \phi''_x(\alpha, x)\,dx = -\phi'_x(\alpha, x) \qquad (\alpha > 0).$$

We have then

$$|\phi'_x(\alpha, x)| = \left|\int_x^\infty \phi''_x(\alpha, x)\,dx\right| \leqq \int_x^\infty |\phi''_x(\alpha, x)|\,dx$$

$$< \int_x^\infty \frac{L}{x^{2+\rho}\alpha^\rho}\,dx = \frac{L}{(1+\rho)x^{1+\rho}\alpha^\rho} < \frac{L}{x^{1+\rho}\alpha^\rho} \qquad (\alpha > 0).$$

Having thus established conditions (g) and (h) we can now prove the following inequalities (50) and (51) which hold when $x = a$. By condition (c) we have

$$|\phi(\alpha, a)| < \frac{N}{a^{2+\rho}\alpha^{2+\rho}} < \frac{N}{a^{2+\rho}} \qquad (\alpha > 1),$$

and by condition (a) we can find a positive constant K_1 such that

$$|\phi(\alpha, a)| < K_1 \qquad (0 \leqq \alpha \leqq 1).$$

Denoting by K the greater of the two quantities $N/a^{2+\rho}$ and K_1 we have

(50) $$|\phi(\alpha, a)| < K \qquad (\alpha \geqq 0).$$

Similarly from conditions (h) and (g) we see that a positive constant C can be found such that

(51) $$|\phi'_x(\alpha, a)| < C \qquad (\alpha \geqq 0).$$

We shall next prove that the integral (48) is absolutely convergent for every positive value of α. By Lemma 1 there exists a positive constant M such that in the interval $x \geqq a$ we have

$$|f(x)| < Mx.$$

Consequently

$$|f(x)\phi(\alpha, x)| < \frac{MN}{x^{1+\rho}\alpha^{2+\rho}}.$$

Since also the integral

$$\int_a^\infty \frac{dx}{x^{1+\rho}}$$

is convergent, the absolute convergence of (48) and the existence of $F(\alpha)$ for values of $\alpha > 0$ follow at once.

We have from (35)

$$F(\alpha) = \int_a^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty [2S'(x) + xS''(x)]\,\phi(\alpha, x)\,dx.$$

Integrating twice by parts we get

$$(52)\quad \begin{aligned} F(\alpha) = [S(x)\phi(\alpha, x)]_a^\infty + [xS'(x)\phi(\alpha, x)]_a^\infty - [S(x)x\phi'_x(\alpha, x)]_a^\infty \\ -\int_a^\infty S(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty S(x)[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx. \end{aligned}$$

Now from the definition of $S(x)$ both $S(a)$ and $S'(a)$ are zero. As x becomes infinite, $S'(x)$ approaches zero by Lemma 6 and $S(x)$ by hypothesis approaches S. From condition (c) we see further that for all positive values of α, $\phi(\alpha, x)$ approaches zero as x becomes infinite.

Combining (49) with conditions (c) and (f) we obtain next

$$\lim_{x=\infty} [x\phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

$$\lim_{x=\infty} [2\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)] = 0 \qquad (\alpha > 0).$$

Hence by Lemma 5

$$\lim_{x=\infty} [x\phi'_x(\alpha, x) + \phi(\alpha, x)] = 0 \qquad (\alpha > 0),$$

from which it follows that

$$\lim_{x=\infty} [x\phi'_x(\alpha, x)] = 0 \qquad (\alpha > 0).$$

The quantities in brackets in equation (52) therefore disappear, and we have

$$(53)\quad F(\alpha) = -\int_a^\infty S(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty S(x)[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx.$$

Put

$$S(x) = S + \epsilon(x)$$

so that

$$\lim_{x=\infty} \epsilon(x) = 0$$

and then substitute in (53). We obtain

$$(54)\quad \begin{aligned} F(\alpha) = -S\int_a^\infty \phi'_x(\alpha, x)\,dx + S\int_a^\infty [\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx \\ -\int_a^\infty \epsilon(x)\phi'_x(\alpha, x)\,dx + \int_a^\infty \epsilon(x)[\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx. \end{aligned}$$

But

$$\int_a^\infty \phi'_x(\alpha, x)\,dx = -\phi(\alpha, a) \qquad (\alpha > 0),$$

$$\int_a^\infty [\phi'_x(\alpha, x) + x\phi''_x(\alpha, x)]\,dx = [x\phi'_x(\alpha, x)]_a^\infty = -a\phi'_x(\alpha, a) \quad (\alpha > 0).$$

Consequently (54) reduces to

$$(55)\qquad F(\alpha) = \phi(\alpha, a)S - a\phi'_x(\alpha, a)S + \int_a^\infty \epsilon(x)x\phi''_x(\alpha, x)\,dx \qquad (\alpha > 0).$$

By condition (d)

$$\phi'_x(0, x) = 0, \qquad \phi''_x(0, x) = 0,$$

and therefore the right hand member of (55) reduces to S when $\alpha = 0$. Since, by condition (b), $\phi''_x(\alpha, x)$ is a continuous function of α for all values of $\alpha \geqq 0$, our theorem is proved if we can show further that the integral in the right hand member of (55) is uniformly convergent in the interval $\alpha \geqq 0$.

Given an arbitrarily small, positive quantity, δ, let us choose $m > a$ and such that

$$(56) \qquad |\epsilon(x)| < \eta = \frac{\rho c^{2+\rho}\delta}{(\rho+1)c^2 L + \rho c^{2+\rho}(aC + K) + \rho N} \qquad (x \geqq m).$$

Consider first values of α in the interval $0 < \alpha$. Let $s = c/\alpha$ and μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered

$$(A) \qquad \mu < s < \nu,$$

$$(B) \qquad \nu \leqq s,$$

$$(C) \qquad \mu \geqq s.$$

Beginning with (A) we write

$$(57) \qquad \int_\mu^\nu \epsilon(x) x \phi''_x(\alpha, x)\,dx = \int_\mu^s + \int_s^\nu = R_1 + R_2.$$

By condition (e) we have

$$|R_1| < \eta \int_\mu^s x\phi''_x(\alpha, x)\,dx \leqq \eta \int_a^s x\phi''_x(\alpha, x)\,dx.$$

But

$$\int_a^s x\phi''_x(\alpha, x)\,dx = s\phi'_x(\alpha, s) - a\phi'_x(\alpha, a) - \phi(\alpha, s) + \phi(\alpha, a)$$

and consequently, in view of (h), (c), (50), and (51),

$$\left|\int_a^s x\phi''_x(\alpha, x)\,dx\right| \leqq |s\phi'_x(\alpha, s)| + |a\phi'_x(\alpha, a)| + |\phi(\alpha, s)| + |\phi(\alpha, a)|$$

$$< \frac{L}{s^\rho \alpha^\rho} + aC + \frac{N}{s^{2+\rho}\alpha^{2+\rho}} + K = \frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K.$$

Hence

$$(58) \qquad |R_1| < \eta\left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K\right].$$

From condition (f) we have for R_2,

$$(59)\qquad \begin{aligned} |R_2| &\leqq \int_s^\nu |\epsilon(x)x\phi_x''(\alpha, x)|\,dx < \eta\int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \\ &= \eta\left[-\frac{L}{\rho x^\rho \alpha^\rho}\right]_s^\nu < \eta\,\frac{L}{\rho c^\rho}. \end{aligned}$$

Combining (57), (58), and (59) and using (56) we have

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &\leqq |R_1| + |R_2| \\ &< \eta\left[\frac{(\rho+1)c^2L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}}\right] = \delta. \end{aligned}$$

For case (B) we have

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &< \eta\int_\mu^\nu x\phi_x''(\alpha, x) \leqq \eta\int_a^s x\phi_x''(\alpha, x)\,dx \\ &< \eta\left[\frac{L}{c^\rho} + aC + \frac{N}{c^{2+\rho}} + K\right] = \eta\left[\frac{\rho c^2 L + \rho c^{2+\rho}(aC+K) + \rho N}{\rho c^{2+\rho}}\right] < \delta. \end{aligned}$$

We have finally for case (C)

$$\begin{aligned} \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| &\leqq \int_\mu^\nu |\epsilon(x)x\phi_x''(\alpha, x)|\,dx \\ &< \eta\int_\mu^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx \leqq \eta\int_s^\nu \frac{L}{x^{1+\rho}\alpha^\rho}\,dx < \eta\,\frac{L}{\rho c^\rho} = \eta\,\frac{c^2L}{\rho c^{2+\rho}} < \delta. \end{aligned}$$

Hence in all three cases

$$(60)\qquad \left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| < \delta \qquad \left(\nu>\mu\geqq m;\ 0<\alpha<\frac{c}{m}\right).$$

For $\alpha = 0$ we have

$$\int_\mu^\nu \epsilon(x)x\phi_x''(0, x)\,dx = \int_\mu^\nu \epsilon(x)\,[\,0\,]\,dx < \delta \qquad (\mu\geqq a).$$

Finally, by taking for q the greater of the two quantities a and m, and by combining the results just obtained we have

$$\left|\int_\mu^\nu \epsilon(x)x\phi_x''(\alpha, x)\,dx\right| < \delta \qquad (\nu>\mu\geqq q;\ \alpha\geqq 0),$$

and our theorem is proved.

We assumed in Theorem IV, for the sake of simplicity of proof, that a was greater than zero. That this assumption really places no restriction upon the generality of the theorem is shown by the following corollary:

COROLLARY. *If, in Theorem* IV, a *is any constant, and the convergence factor* $\phi(\alpha, x)$ *satisfies conditions* (a), (b), (c), (d), (e), *and* (f) *for* $x \geqq b$, *where* b *is any positive constant, and conditions* (a) *and* (d) *for* $a \leqq x \leqq b$, *the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = A.$$

We have

$$\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \frac{1}{x}\int_a^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x\int_a^b f(\beta)\,d\beta d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^x A\,d\alpha$$

$$= \frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + \frac{1}{x}\int_a^b [\psi(\alpha) - \psi(b)]\,d\alpha + \frac{x-a}{x}A,$$

and consequently

$$S = \lim_{x=\infty}\frac{1}{x}\int_a^x\int_a^\alpha f(\beta)\,d\beta d\alpha = \lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha + A$$

or

$$\lim_{x=\infty}\frac{1}{x}\int_b^x\int_b^\alpha f(\beta)\,d\beta d\alpha = S - A.$$

Hence the integral

$$\int_b^\infty f(x)\,dx$$

is summable and has the value $(S - A)$. Therefore by Theorem IV the integral

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely convergent for all positive values of α, its value $G(\alpha)$ is continuous for such values, and

$$\lim_{\alpha=+0} G(\alpha) = S - A.$$

Hence

$$\int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx \tag{61}$$

is absolutely convergent for all positive values of α.

Let

$$\int_a^b f(x)\phi(\alpha, x)\,dx = H(\alpha).$$

Then

$$H(0) = \int_a^b f(x)\phi(0, x)\,dx = \int_a^b f(x)\,dx = A.$$

We have by (61)

$$F(\alpha)=\int_a^\infty f(x)\phi(\alpha, x)\,dx=H(\alpha)+G(\alpha),$$

and since $H(\alpha)$ and $G(\alpha)$ are continuous for all values of $\alpha>0$, $F(\alpha)$ is continuous for such values. Moreover, since $H(\alpha)$ and $G(\alpha)$ each approach a limit when $\alpha=+0$, so also does $F(\alpha)$ and we have for that limit

$$\lim_{\alpha=+0} F(\alpha)=H(0)+\lim_{\alpha=+0} G(\alpha)=S.$$

§ 5. *Convergence factors in convergent integrals.*

Theorem IV is applicable to convergent integrals since by Theorem III every convergent integral is summable. As in the case of series, however, less restriction can be placed upon the convergence factor, as is shown in the following theorem:

THEOREM V. *If the function $f(x)$ is finite and integrable in every finite interval lying in the interval $x \geqq a$, where a is any positive constant, and the integral*

$$\int_a^\infty f(x)\,dx \tag{62}$$

converges to the value A, then the integral

$$\int_a^\infty f(x)\phi(\alpha, x)\,dx \tag{63}$$

will be absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore will define a continuous function, provided the convergence factor $\phi(\alpha, x)$ satisfies the following conditions for $x \geqq a$:*

(*a*) $\qquad \phi(\alpha, x)$ *is continuous in* (α, x) $\qquad (\alpha \geqq 0)$,

(*b*) $\qquad \phi'_x(\alpha, x)$ *exists* $\qquad (\alpha \geqq 0)$,

(*c*) $\qquad |\phi(\alpha, x)| < \dfrac{N}{x^{1+\rho}\alpha^{1+\rho}}$ $\qquad (\alpha > 0)$,

* That the integral (63) converges more rapidly than (62) can be shown by a method precisely analogous to that suggested for series in a previous footnote (see page 307) since for a given value of α, an m can be determined such that

$$\left|\int_n^\infty f(x)\phi(\alpha, x)\,dx\right| < \left|\int_n^\infty f(x)\,dx\right| \qquad (n \geqq m).$$

$$(d) \qquad \phi(0, x) = 1,$$

$$(e) \qquad \phi'_x(\alpha, x) \leqq 0 \, * \qquad (0 \leqq x\alpha \leqq c),$$

$$(f) \qquad |\phi'_x(\alpha, x)| < \frac{L}{x^{1+\rho}\alpha^{\rho}} \qquad (\alpha > 0),$$

where N, ρ, c, and L are positive constants.

We must first derive from the given conditions the following further condition which holds for $x \geqq a$:

$$(g) \qquad |\phi(\alpha, x)| < K \qquad (\alpha \geqq 0),$$

where K is a positive constant.

We know from condition (e) † that $\phi(\alpha, x)$, for a given value of α, either decreases or remains constant with increasing x as long as $x\alpha$ does not exceed c. We have then

$$(64) \qquad \phi(\alpha, a) \geqq \phi(\alpha, x) \geqq \phi\left(\alpha, \frac{c}{\alpha}\right) \qquad (0 < x\alpha < c).$$

But by condition (a) we can find a positive constant M such that

$$|\phi(\alpha, a)| < M \qquad (0 < \alpha < c/a),$$

and from condition (c)

$$\left|\phi\left(\alpha, \frac{c}{\alpha}\right)\right| < \frac{N}{c^{1+\rho}},$$

so that if we take as K_1 the greater of the two quantities M and $N/c^{1+\rho}$ we have by (64)

$$(65) \qquad |\phi(\alpha, x)| < K_1 \qquad (0 < x\alpha < c).$$

For $x\alpha \geqq c$ we have from condition (c)

$$(66) \qquad |\phi(\alpha, x)| \leqq \frac{N}{c^{1+\rho}} \leqq K_1 \qquad (x\alpha \geqq c).$$

For $\alpha = 0$ we have from condition (d)

$$(67) \qquad \phi(0, x) = 1.$$

* This is the only condition that is not satisfied by the convergence factor of Theorem IV. We may substitute for it the condition

$$(e') \qquad \phi'_x(\alpha, x) \geqq 0 \qquad (0 \leqq x\alpha \leqq c).$$

If (e) $\big($or $(e')\big)$ holds for all values of x and α, (f) is unnecessary.

† If we substitute (e') for (e), the inequality signs in (64) are reversed but this does not affect the reasoning.

Hence if we choose K greater than either of the quantities K_1 and 1, we have from (65), (66), and (67)

$$\phi(\alpha, x) < K \qquad (\alpha \geqq 0).$$

Let δ be an arbitrarily small positive quantity. Since $\phi(\alpha, x)$ by condition (a) is a continuous function of α for all values of $\alpha \geqq 0$, our theorem will be proved if we can show that a positive quantity q exists such that

$$(68) \qquad \left|\int_\mu^\nu f(x)\phi(\alpha, x)\,dx\right| < \delta \qquad (\nu > \mu \geqq q;\ \alpha \geqq 0).$$

Determine $m > a$ and such that

$$(69) \qquad \left|\int_\mu^\nu f(x)\,dx\right| < \eta = \frac{\rho c^\rho \delta}{L + 3K\rho c^\rho} \qquad (\nu > \mu \geqq m).$$

We first consider values of $\alpha > 0$. Integrating by parts we get

$$(70) \qquad \begin{aligned} \int_\mu^\nu f(x)\phi(\alpha, x)\,dx &= \left[\phi(\alpha, x)\int_\mu^x f(x)\,dx\right]_\mu^\nu - \int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx \\ &= \phi(\alpha, \nu)\int_\mu^\nu f(x)\,dx - \int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx. \end{aligned}$$

Let $s = c/\alpha$ and let μ and ν be any two quantities such that

$$m \leqq \mu < \nu.$$

Three cases must be considered:

(A) $\qquad \mu < s < \nu,$

(B) $\qquad \nu \leqq s,$

(C) $\qquad \mu \geqq s.$

Beginning with (A) we write

$$(71) \qquad \int_\mu^\nu\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\,dx = \int_\mu^s + \int_s^\nu = R_1 + R_2.$$

By virtue of (69) and conditions (e) and (g) we have for R_1

$$(72) \qquad \begin{aligned} |R_1| &\leqq \int_\mu^s \left|\left[\int_\mu^x f(x)\,dx\right]\phi_x'(\alpha, x)\right| dx < \eta\int_\mu^s [-\phi_x'(\alpha, x)]\,dx \\ &= \eta\,[\phi(\alpha, \mu) - \phi(\alpha, s)] \leqq \eta\,|\phi(\alpha, \mu)| + \eta\,|\phi(\alpha, s)| < 2K\eta \\ &\qquad \left(\nu > \mu \geqq m;\ 0 < \alpha < \frac{c}{m}\right). \end{aligned}$$

From condition (f) we have for R_2

$$
\begin{aligned}
|R_2| &\leqq \int_s^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi_x'(\alpha, x) \right| dx < \eta \int_s^\nu \frac{L}{x^{1+\rho}\,\alpha^\rho}\,dx \\
&= \frac{\eta L}{\alpha^\rho}\left[-\frac{1}{\rho x^\rho} \right]_s^\nu < \frac{\eta L}{\rho(s\alpha)^\rho} = \frac{\eta L}{\rho c^\rho}.
\end{aligned} \tag{73}
$$

From (70)–(73) we get

$$
\begin{aligned}
\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| &\leqq \left| \phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx \right| + |R_1| + |R_2| \\
&< K\eta + 2K\eta + \frac{\eta L}{\rho c^\rho} = \eta \frac{L + 3K\rho c^\rho}{\rho c^\rho} = \delta.
\end{aligned}
$$

For case (B) we have

$$
\begin{aligned}
\left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi_x'(\alpha, x)\,dx \right| &\leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi_x'(\alpha, x) \right| dx \\
&< \eta \int_\mu^s [-\phi_x'(\alpha, x)]\,dx = \eta [\phi(\alpha, \mu) - \phi(\alpha, s)] < 2K\eta,
\end{aligned}
$$

which combined with (70) gives

$$
\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| < \left| \phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx \right| + 2K\eta < 3K\eta = \eta \frac{3K\rho c^\rho}{\rho c^\rho} < \delta.
$$

Finally we have for case (C)

$$
\begin{aligned}
\left| \int_\mu^\nu \left[\int_\mu^x f(x)\,dx \right] \phi_x'(\alpha, x)\,dx \right| &\leqq \int_\mu^\nu \left| \left[\int_\mu^x f(x)\,dx \right] \phi_x'(\alpha, x) \right| dx \\
&< \eta \int_\mu^\nu |\phi_x'(\alpha, x)|\,dx < \eta \int_\mu^\nu \frac{L}{x^{1+\rho}\,\alpha^\rho}\,dx \leqq \eta \int_s^\nu \frac{L}{x^{1+\rho}\,\alpha^\rho}\,dx < \eta \frac{L}{\rho c^\rho},
\end{aligned}
$$

which combined with (70) gives

$$
\begin{aligned}
\left| \int_\mu^\nu f(x)\phi(\alpha, x)\,dx \right| &< \left| \phi(\alpha, \nu) \int_\mu^\nu f(x)\,dx \right| + \eta \frac{L}{\rho c^\rho} \\
&< \eta \left[K + \frac{L}{\rho c^\rho} \right] = \eta \frac{L + K\rho c^\rho}{\rho c^\rho} < \delta.
\end{aligned}
$$

Hence in all three cases we have established the inequality (68), q being taken equal to m.

For $\alpha = 0$ the integral (63) becomes the integral (62), and since the latter is convergent, we can choose m_1 such that

$$
\left| \int_\mu^x f(x)\phi(0, x)\,dx \right| = \left| \int_\mu^x f(x)\,dx \right| < \delta \qquad (\nu > \mu \geqq m_1).
$$

To establish the inequality (68) we have now only to take q equal to the greater of the two quantities m and m_1.

Theorem V has been proved under the assumption that a is greater than zero. As in the treatment of summable integrals, however, the theorem can easily be extended to the case where a is any constant. We shall show this by proving the following corollary:

Corollary. *If, in Theorem* V, *a is any constant and the convergence factor $\phi(\alpha, x)$ satisfies conditions (a), (b), (c), (d), (e), and (f) for $x \geqq b$, where b is any positive constant, and conditions (a) and (d) for $a \leqq x \leqq b$, the theorem holds without change.*

Let

$$\int_a^b f(x)\,dx = B.$$

We have

$$A = \int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

and consequently

$$\int_b^\infty f(x)\,dx = A - B.$$

Hence by Theorem V

$$\int_b^\infty f(x)\phi(\alpha, x)\,dx = G(\alpha)$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

Then

$$\int_a^b f(x)\phi(\alpha, x)\,dx + \int_b^\infty f(x)\phi(\alpha, x)\,dx = \int_a^\infty f(x)\phi(\alpha, x)\,dx$$

is absolutely and uniformly convergent for all values of $\alpha \geqq 0$ and therefore defines a continuous function for such values.

The convergence factor $e^{-x\alpha}$ satisfies the conditions of Theorem V. Hence if the integral

$$\int_a^\infty e^{kx} f(x)\,dx$$

is convergent, the integral

$$\int_a^\infty e^{(k-\alpha)x} f(x)\,dx$$

converges uniformly in the interval $\alpha \geqq 0$. If now we set

$$e^{(k-\alpha)} = m, \qquad e^k = m_1,$$

we have the theorem:

If the integral

$$\int_a^\infty m_1^x f(x)\,dx$$

is convergent, then the integral

$$\int_a^\infty m^x f(x)\,dx$$

converges uniformly in the interval $0 \leqq m \leqq m_1$.

This theorem is due to BONNET (cf. Liouville's Journal, vol. 14 (1849), p. 250).

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